Semigroups and Normal Categories

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Introduction

The theme of this lecture notes is the theory of normal categories and its relation with structure theory of of regular semigroups. The concept of normal categories was introduced by KSS Nambooripad (cf.[8]) in developing the cross connection related structure theory of regular semigroups. Various aspects of normal categories and its relation to semigroups are discussed here. Familiarity with the elemntary concepts in semigroup theory and category theory helps in following the discussions.

1 Categories

The general terminology in category theory used are as follows.

- A category is a pair $(v\mathcal{C},\ m\mathcal{C})$ where $v\mathcal{C}$ is called the class of objects and $m\mathcal{C}$ the class of morphisms.
- A category $(v\mathcal{C},\ m\mathcal{C})$ is said to be a small category if $v\mathcal{C}$ is a set. It will follow that in this case $m\mathcal{C}$ is also a set.
- With each $f \in m\mathcal{C}$ is associated two objects a, called the domain of f, and b, called the codomain of f. We denote this relation by writing $f : a \to b$.
- We denote the set of all morphisms with domain a and codomain b as

$$m(a,b) \text{ or } [a,b]_{\mathcal{C}} \text{ or } hom(a,b).$$

- There is a composition $m(a,b) \times m(b,c) \to m(a,c)$ such that

$$(f,g) \mapsto fg.$$

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[dr, "fg"'] & b \arrow[d, "g"] \\
& c
\end{tikzcd}
$$

The composition satisfies the following.

(i) $f(gh) = (fg)h$, whenever the products are defined.

(ii) Every morphism $f \in m(a, b)$ has a right identity 1_b and a left identity 1_a such that

$$f1_b = f \text{ and } 1_a f = f.$$

We usually denote a category $(v\mathcal{C}\ , m\mathcal{C})$ by $\mathcal{C}$ only. Also we write $\mathcal{C}$ to denote the morphism class $m\mathcal{C}$ so that $f \in \mathcal{C}$ means that f is a morphism in $\mathcal{C}$.

Example 1 (Category of subsets)**.** *An easily identified example of a category is the category $C(X)$ of all subsets of a set X.*

Here morphisms are mappings between the sets and the composition is usual composition of mappings.
That is, if

$$f : A \to B \textit{ and } g : B \to C \textit{ then } fg : A \to C.$$

Clearly the identity maps

$$1_A : A \to A \textit{ and } 1_B : B \to B$$

are such that

$$1_A f = f = f1_B$$

for all $f : A \to B$.

A similar category is the category of subspaces of a vector space.

Example 2 (Category of subspaces)**.** *Let V be a vector space. Then the category of subspaces of V is the category $C(V)$ whose objects are the subspaces of V.*

Here morphisms are linear mappings between the subspaces.

In several contexts categories are presented by simply specifying the morphisms. Here the identity morphisms serve the role of objects. As an example we see that any group can be realised as a category.

Example 3 (Groups as ctegories)**.** *Let G be a group. Then the elements of G can be realised as morphisms and the identity of G in place of object.*

Thus every group is a category having only one object.

Relations such as equivalence relations, partial orders etc. can also be realized as categories.

Example 4 (Partially ordered sets)**.** *Let $(X, \leq)$ be a partially ordered set. Then X is a category whose objects are elements of X and morphisms are defined as follows. For $x, y \in X$ the morphism set $[x, y]$ is given by*

$$[x,y] = \begin{cases} (x,y) \text{ if } x \leq y \\ \emptyset \text{ otherwise.} \end{cases}$$

Here composition is given by

$$(x, y)(y, z) = (x, z).$$

The transitivity justifies the definition of composition.

Clearly reflexivity gives that (x, x) is a morphism in the category for every x and thsese are the identity morphisms.

A class of categories called **preorders** is defined as follows.

Example 5 (Preorders)**.** *A small category $\mathcal{C}$ is said to be a* **preorder** *if for any $a, b \in v\mathcal{C}$, $[a, b]_{\mathcal{C}}$ is either singleton or empty.*

In this case we can define a relation $\leq$ on $v\mathcal{C}$ as follows

$$a \leq b \text{ if } [a, b]_{\mathcal{C}} \text{ is nonempty.}$$

We see that $\leq$ is a reflexive and transitive relation on $v\mathcal{C}$. Such relations are called **quasi orders.**

Thus every preorder provides a quasi order on the object set. Conversely evrey quasi order can be realised as a category which is a preoreder.

The following theorem connects partial orders with preorders. Here we identify special morphisms in a category called isomorphism. Isomorphisms are just the invertible morphisms in a category. That is, $f : a \to b$ in $\mathcal{C}$ is an isomorphism if there is $g : b \to a$ in $\mathcal{C}$ such that

$$fg = 1_a \text{ and } gf = 1_b.$$

Theorem 1. *Let $\mathcal{C}$ be a preorder. Then the induced quasi order $\leq$ is a partial order if and only if all isomorphisms in $\mathcal{C}$ are identities.*

The statement "isomorphisms are identities" is equivalent to the anti symmetry of the quasi order $\leq$ and so $\leq$ becomes a partial order.

1.1 Monomorphism and epimporphism

Morphisms which resemble the properties of one to one and onto mappings are described as monomorphisms and epimorphisms respectively.

Definition 1. *A morphism $f : a \to b$ is said to be a* **monomorphism** *if the following hold.*
For any morphisms $g, h : c \to a$ for $c \in v\mathcal{C}$

$$\text{if } gf = hf \text{ then } g = h.$$

A morphism $f : a \to b$ is said to be an **epimorphism** *if the following hold.*
For any morphisms $g, h : b \to c$ for $c \in v\mathcal{C}$

$$\text{if } fg = fh \text{ then } g = h.$$

Note that in categories of sets and mappings monomorphisms are exactly the one to one mappings and epimorphisms are the onto mappings.

Further bijective maps are both mono and epi. But a mono and epi morphism may not be an isomorphism in certain categories. For example in category of topological spaces with continuous maps as morphisms, isomorphisms are homeomorphisms. Here a continuous and bijective map can fail to be a homeomorphism.

Now we state an alternate definition for small categories in terms of the morphisms. This description suggests that small categories can be realised as algebraic structures having a partial binary opertaion on it. This way it is a generalization of groups as well as certain semigroups.

Here the set $m\mathcal{C}$ is denoted by $\mathcal{C}$ and the category structure is essentially that of a set with a partial binary operation. A partial binary operation on a set X is a map

$$\circ : D_X \to X$$

where $D_X \subseteq X \times X$ is called the domain of the partial binary operation.

Definition 2. *A small category is a triple $(\mathcal{C}, D_{\mathcal{C}}, \circ)$ where $\circ$ is a partial binary operation on $\mathcal{C}$ and $D_{\mathcal{C}}$ is the domain of the partial binary operation with the following properties.*

(i) For every x in $\mathcal{C}$ there is a unique e_x and a unique f_x in $\mathcal{C}$ such that

$$e_x \circ e_x = e_x \text{ and } e_x \circ x = x$$

and

$$f_x \circ f_x = f_x \text{ and } x \circ f_x = x.$$

e_x and f_x are called the left and right identities respectively of f in $\mathcal{C}$. In the following $\circ$ will be replaced by $\cdot$ or by juxtaposition.

(ii) $D_{\mathcal{C}} = \{(x, y) \in \mathcal{C} \times \mathcal{C} : f_x = e_y\}$.

(iii) *The partial composition* $\circ$ *is associative. That is, for* $x, y, z \in \mathcal{C}$

$$x(yz) = (xy)z$$

whenever the products are defined.

1.2 Functors and Natural transformations

Category theory is sometimes referred to as theory of functors and natural transformations. Functors are structure preserving mappings between categories and natural transformations provide certain connections between functors.

Definition 3. *Let* $\mathcal{C}, \mathcal{D}$ *be categories. A functor* F *from* $\mathcal{C}$ *to* $\mathcal{D}$ *is a pair of mappings, one from* $v\mathcal{C}$ *to* $v\mathcal{D}$ *and the other from* $m\mathcal{C}$ *to* $m\mathcal{D}$ *(both denoted by* F *for convenience) satisfying the following.*

- *If* $f : a \to b$ *in* $\mathcal{C}$ *then* $F(f) : F(a) \to F(b)$ *in* $\mathcal{D}$.
- *For each* $a \in v\mathcal{C}$
$$F(1_a) = 1_{F(a)}.$$
- *For* $f : a \to b$ *and* $g : b \to c$ *in* $\mathcal{C}$
$$F(fg) = F(f)F(g).$$

Some natural examples are the following.

Example 6 (Free groups)**.** *Let* X *be a nonempty set and* $F(X)$ *be the free group on* X*. Now if* Y *is any set and* $f : X \to Y$ *is a map then* f *induces a homomorphism from* $F(X)$ *to* $F(Y)$*. We may consider this homomorphism as* $F(f)$*. Then* F *is a functor from the category of sets to the category of groups.*

The next example is often used in applications of categories.

Example 7. *Let* $\mathcal{C}$ *be any category and for* $a, b \in v\mathcal{C}$ *let* $Hom(a, b)$ *denote the set of morphisms from* a *to* b*. For each* $a \in v\mathcal{C}$ *we define a functor* $Hom(a, -)$ *from* $\mathcal{C}$ *to the category* $\mathcal{S}et$ *of sets and mappings as follows.*
For $c, d \in v\mathcal{C}$ *and* $f : c \to d$

$$Hom(a, -)(c) = Hom(a, c) = [a, c]_{\mathcal{C}}$$

and $Hom(a,-)(f) = Hom(a,f) : Hom(a,c) \to Hom(a,d)$ *is defined by*

$$h \mapsto hf$$

for all $h : a \to c$. *These functors are called homfunctors.*

Now we define natural transformations. These are relations between functors described as follows.

Definition 4. *Let* $\mathcal{C}, \mathcal{D}$ *be categories and* F, G *be functors from* $\mathcal{C}$ *to* $\mathcal{D}$. *A natural transformation* $\eta : F \to G$ *is a collection* $\{\eta_a : a \in v\mathcal{C}\}$ *of morphisms in* $\mathcal{D}$ *such that the following hold.*

(i) For each $a \in v\mathcal{C}, \eta_a$ *is from* $F(a)$ *to* $G(a)$.

(ii) For $f : a \to b$ *in* $\mathcal{C}$

$$\eta_a G(f) = F(f)\eta_b.$$

That is the following diagram is commutative.

$$\begin{tikzcd} F(a) \arrow[r, "\eta_a"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\eta_b"'] & G(b) \end{tikzcd}$$

Example 8. *There is a naturally defined natural transformation between homfunctors. Let* $\mathcal{C}$ *be a category and* $Hom(a,-)$ *and* $Hom(b,-)$ *be homfunctors and* $g : b \to a$ *be a morphism.*
Then $Hom(g,-) : Hom(a,-) \to Hom(b,-)$ *is a natural transformation defined by*

$$Hom(g,-)_c = Hom(g,c) : Hom(a,c) \to Hom(b,c)$$

which maps

$$h \mapsto gh.$$

It may be seen that for $k : c \to d$ *in* $\mathcal{C}$ *the following diagram is commutative. We often write* $H(a,b)$ *in place of* $Hom(a,b)$.

$$\begin{array}{ccc} H(a,c) & \xrightarrow{H(g,c)} & H(b,c) \\ {\scriptstyle H(a,k)}\downarrow & & \downarrow{\scriptstyle H(b,k)} \\ H(a,d) & \xrightarrow[H(g,d)]{} & H(b,d) \end{array}$$

The following theorem describes all natural transformations between homfunctors.

Theorem 2 (cf. Yoneda Lemma)**.** *Let $\mathcal{C}$ be a category and $a, b \in v\mathcal{C}$. Let $\eta : Hom(a, -) \to Hom(b, -)$ be a natural transformation. Then there exists a morphism $g : b \to a$ such that $\eta = Hom(g, -)$.*

2 Normal Category

The concept of normal categories was introduced by KSS Nambooripad in his structure theory of regular semigroups in terms of cross connections. A normal category is an abstraction of the category of principal left ideals of a regular semigroup. The role of principal left ideals can be replaced ny principal right ideals aswell.

The essential features of a normal category are the following.

- A partial order on the object set.
- A specified morphism called inclusion morphism $j(a, b) : a \to b$ whenever $a \leq b$.
- A cluster of morphisms called normal cones
- A specific factorization of morphisms called normal factorization.

We begin by introducing the concept of category with normal factorization.

Definition 5 (Category with normal factorization)**.** *A category with normal factorization is a small category $\mathcal{C}$ with the following properties.*

(i) The vertex set $v\mathcal{C}$ of $\mathcal{C}$ is a partially ordered set such that whenever $a \leq b$ in $v\mathcal{C}$, there is an associated monomorphism $j(a, b) : a \to b$ satisfying the following.

(a) *For all* $a \in v\mathcal{C}$, $j(a,a) = 1_a$.

(b) *For* $a, b, c \in v\mathcal{C}$ *with* $a \le b \le c$

$$j(a,b)j(b,c) = j(a,c).$$

(c) *For* $a, b \le c$ *in* $v\mathcal{C}$, *if*

$$j(a,c) = fj(b,c)$$

for some $f : a \to b$ *then* $a \le b$ *and* $f = j(a,b)$.

(d) *Every morphism* $j(a,b) : a \to b$ *has a right inverse* $q : b \to a$ *such that*

$$j(a,b)q = 1_a.$$

Such a morphism q *is called a retraction in* $\mathcal{C}$.

(ii) *Every morphism* f *in* $\mathcal{C}$ *has a factorization*

$$f = quj$$

where q *is a retraction,* u *is an isomorphism and* j *is an inclusion. Such a factorization is called normal factorization in* $\mathcal{C}$.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "q"'] & b \\ a_0 \arrow[r, "u"'] & b_0 \arrow[u, "j"'] \end{tikzcd}$$

Here $a_0 \le a$ *and* $b_0 \le b$.

The normal factorization of a morphism is not in general unique. But there is an invariant component which is called the epimorpic component of the morphism.

Proposition 1 (Epimorphic component and Image)**.** *Let* $\mathcal{C}$ *be a category with normal factorization. If*

$$f = quj \text{ and } f = q'u'j'$$

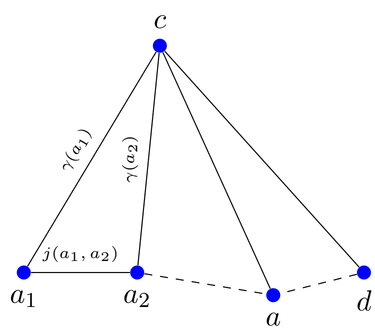
are two normal factorizations of $f \in \mathcal{C}$ *then* $j = j'$ *and* $qu = q'u'$.

- In this case $f^\circ = qu$ is called the epimorphic component of f.
- The codomain of f° is called the image of f and is denoted by $\operatorname{Im} f$

Another feature of normal categories is the existence of clusters of morphisms called normal cones. A normal cone is defined as follows.

Definition 6 (Normal Cones)**.** *A* cone γ *with vertex* c *is a function from* $v\mathcal{C}$ *to* $m\mathcal{C}$*, satisfying the following*

- $\gamma(a) \in \mathcal{C}(a, c)$ *for all* $a \in v\mathcal{C}$*.*
- *If* $a_1 \subseteq a_2$ *then* $\gamma(a_1) = j(a_1, a_2)\gamma(a_2)$



A cone γ *is said to be a* **normal cone** *if there exists* $d \in v\mathcal{C}$ *such that* $\gamma(d)$ *is an isomorphism.*

Remark 1. *In a normal cone there can be more than one* d *such that* $\gamma(d)$ *is an isomorphism. The set*

$$M\gamma = \{d \in v\mathcal{C} : \gamma(d) \text{ is an isomorphism}\}$$

is called the M*-set of the normal cone* γ*. This set will be useful in describing special features of normal categories. It may be noted that* $M\gamma$ *is nonempty for every normal cone* γ*.*

Now we define normal categories.

Definition 7 (Normal Categories)**.** *A normal category is a category* $\mathcal{C}$ *with normal factorization such that for each* $a \in v\mathcal{C}$ *there is a normal cone* γ *with vertex* a *such that* $\gamma(a) = 1_a$*.*

In [8] a normal category is defined as a category with normal factorization in which for every object c there exists a normal cone γ such that $\gamma(c)$ is an isomorphism. Now we show that both these descriptions are equivalent.

Proposition 2. *Let $\mathcal{C}$ be a category with normal factorization. Then the following are equivalent.*

(i) For each $a \in v\mathcal{C}$ there is a normal cone γ with vertex a such that $\gamma(a) = 1_a$.

(ii) For each $a \in v\mathcal{C}$ there is a normal cone γ such that $\gamma(a)$ is an isomorphism.

Proof. Clearly $(i) \implies (ii)$. So it is sufficient to show that whenever $\gamma(a)$ is an isomorphism there is a normal cone σ with vertex a such that $\sigma(a) = 1_a$. Towards this we describe σ as follows. For each $c \in v\mathcal{C}$ define

$$\sigma(c) = \gamma(c)(\gamma(a))^{-1}.$$

Then $\sigma(c) : c \to a$ for all $c \in v\mathcal{C}$. It is easy to see that σ is a normal cone with vertex a. Now

$$\sigma(a) = \gamma(a)(\gamma(a))^{-1} = 1_a.$$

□

Now we see some examples of normal categories.

Example 9 (Category of subsets)**.** *The category $C(X)$ of all subsets of a set X is easily seen to be a normal category. Here the partial order on objects is the usual inclusion of sets. Also the inclusion morphisms are the usual inclusion maps.*

We show that every inclusion $j : A \to B$ has a right inverse where $A \subseteq B$. Fix an element $t \in A$. Define $q : B \to A$ by

$$q(x) = \begin{cases} x \text{ if } x \in A \\ t \text{ if } x \in B \text{ and } x \notin A. \end{cases}$$

Clealy

$$jq = 1_A.$$

Thus every inclusion has a right inverse. We may verify the remaining axioms.

For $A, B, C \in vC(X)$ let $A, B \leq C$ and $f : A \to B$ be such that

$$j(A, C) = fj(B, C).$$

Now for any $x \in A$

$$(x)j(A, C) = x \text{ and } (x)fj(B, C) = (x)f.$$

So $(x)f = x$ for all $x \in A$.

That is $A \subseteq B$ and f is an inclusion.

Normal factorization is easy to see. For if $f : A \to B$ is a map then Choose B_0 to be the image of f and A_0 to be a cross section of

$$ker f = \{(x, y) : f(x) = f(y)\}.$$

Then $A_0 \leq A$ and $B_0 \leq B$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ A_0 \arrow[u, "j"] & B_0 \arrow[u, "j"'] \end{tikzcd}$$

Now $q : A \to A_0$ is defined by

$$q(x) = x_0 \text{ where } x_0 \in A_0 \text{ is such that } f(x) = f(x_0).$$

Clearly q is a retraction. Let u be the restriction of f to A_0 and $j = j(B_0, B)$. Then

$$f = quj$$

and this is a normal factorization of f.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "q"'] & B \\ A_0 \arrow[r, "u"'] & B_0 \arrow[u, "j"'] \end{tikzcd}$$

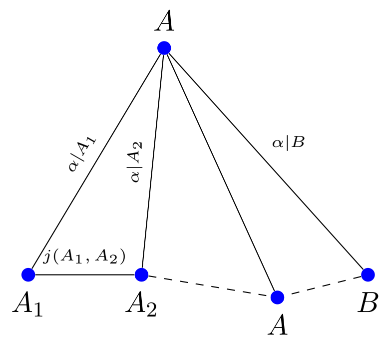
It may be noted that here A_0 and q have several possible choices and thus the factorization is not unique.

But B_0 is the image of f and so is fixed by f. Consequently the j in the factorization is unique.

To conclude that $C(X)$ is a normal category it remains to show that normal cones as required are available. For any $A \in vC(X)$ consider a map $\alpha : X \to A$ which is onto. Now for each $B \in vC(X)$ define $\gamma(B)$ to be the restriction of α to B, that is

$$\gamma(B) = \alpha|B \text{ for all } B \in vC(X).$$

We can see that γ is a normal cone in $C(X)$ with vertex A.



Choosing α to be an extension of the identity map on A to a map from X to A we see that the induced normal cone γ has the property that

$$\gamma(A) = 1_A.$$

Thus $C(X)$ is a normal category.

In this example $X \in vC(X)$ and so $C(X)$ has a greatest object X. We may exclude X from the set of objects and still get a normal category. as in the following.

Example 10. *Consider the category $C_0(X)$ of all proper subsets of a non empty set X. Now $C_0(X)$ is a subcategory of $C(X)$ and it is easy to see that $C_0(X)$ is also a normal category with the same inclusions and retractions as in $C(X)$.*

Another interesting example of normal category is the category of subspaces of a finite dimensional vector space.

Example 11. *Let V be a finite dimensional vector space and $\mathcal{C}(V)$ be the category of all subspaces of V with morphisms as linear maps. Then inclusion is a partial order on $v\mathcal{C}(V)$ and inclusion maps are linear. With the usual partial order and inclusions $\mathcal{C}(V)$ becomes a normal category.*

The verifications are similar to the one in the case of $C(X)$. Still we provide the details here.

To see that every inclusion $j : A \to B$ has a right inverse consider subspaces A, B of V such that $A \subseteq B$. Find a subspace A' of B such that

$$A \oplus A' = B.$$

Then every element b of B has a unique representation

$$b = a + a' \text{ for some } a \in A \text{ and } a' \in A'.$$

Then the map $q : B \to A$ defined by

$$q(b) = a$$

is a linear map and is a retraction from B to A. Thus every inclusion has a right inverse.

Towards the remaining property of inclusions consider subspaces A, B, C of V and let $A, B \subseteq C$ and $f : A \to B$ be a linear map such that such that

$$j(A, C) = fj(B, C).$$

Now for any $x \in A$

$$(x)j(A, C) = x \text{ and } (x)fj(B, C) = (x)f.$$

So $(x)f = x$ for all $x \in A$. That is $A \subseteq B$ and f is an inclusion.

To see normal factorizations consider a linear map $f : A \to B$. Let B_0 be the image of f and N be the null space of f. Let A_0 be a subspace of A such that

$$A = N \oplus A_0.$$

Then $A_0 \leq A,\ B_0 \leq B$ and

$$u = f|A_0 : A_0 \to B_0$$

is an isomorphism.

Now $q : A \to A_0$ be the retraction from A to A_0 defoned by

$$q(a) = a_0 \text{where } a = a_0 + n$$

for the uniquely determined $a_0 \in A_0$ and $n \in N$. Then

$$f = quj$$

and this is a normal factorization of f.

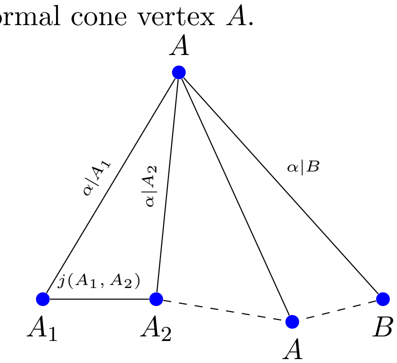
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "q"'] & B \\
A_0 \arrow[r, "u"'] & B_0 \arrow[u, "j"']
\end{tikzcd}$$

Now it remains to show that normal cones as required are available.

For any $A \in v\mathcal{C}(V)$ consider a linear map $\alpha : V \to A$ which is onto. Defining $\gamma(B)$ to be the restriction of α to B, that is

$$\gamma(B) = \alpha|B \text{ for all } B \in v\mathcal{C}(V).$$

We see that γ is a normal cone vertex A.



Choosing α to be a projection on A we see that the induced normal cone γ has the property that

$$\gamma(A) = 1_A.$$

Thus $\mathcal{C}(V)$ is a normal category.

Here also we see that $v\mathcal{C}(V)$ has a greatest object and we may consider the category $\mathcal{C}_0(V)$ of all proper subspaces of V which also is a normal category and $\mathcal{C}_0(V)$ has no greatest object.

Remark 2. *In categories with maximum objects all normal cones are easily determined.For example if m is a maximum object in $\mathcal{C}$ then every normal cone γ in $\mathcal{C}$ is completely determined by $\gamma(m)$. In this case for any $c \in v\mathcal{C}$ we have*

$$\gamma(c) = j(c.m)\gamma(m).$$

The next example is a category with three objects.

Example 12. *Let $\mathcal{C}$ be the category with the following three sets as objects.*

$$A = \{x, y, z\}, B = \{x, y, w\} \text{ and } C = A \cap B = \{x, y\}.$$

Here morphisms are maps f such that $f(C) = C$.

For example there are only six morphisms in $[A, B]_{\mathcal{C}}$ given by

$$\{f_1, f_2, f_3, g_1, g_2, g_3\}$$

where

$$f_1|C = f_2|C = f_3|C = 1_C \text{ and } f_1(z) = x, f_2(z) = y, f_3(z) = w$$

and

$$g_1|C = g_2|C = g_3|C = t \text{ and } g_1(z) = x, g_2(z) = y, g_3(z) = w$$

where t is the transposition (x, y). Similarly other morphism sets can be found.

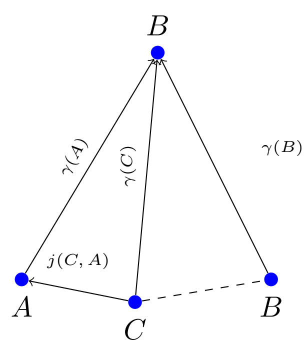
It is easily seen that $\mathcal{C}$ is a category with normal factorization. Now we describe the normal cones here.

Let γ be a normal cone with vertex B and let

$$\gamma(A) = f_1.$$

Since f_1 is not an isomorphism we need $\gamma(B)$ to be an isomorphism. It is easy to see that there are only two isomorphisms $\{1_B, h_1\}$ from B to B where $h_1|C$ is the transposition. It follows that

$$\gamma(B) = 1_B.$$



It can be seen that there are three normal cones $\gamma_1, \gamma_2, \gamma_3$ with vertex B and

$$\gamma_i(B) = 1_B.$$

These are prescribed by

$$\gamma_i(A) = f_i \text{ for } i = 1, 2, 3.$$

Similarly we see that there are three normal cones $\delta_1, \delta_2, \delta_3$ with vertex B and

$$\delta_i(B) = \beta$$

where β is the isomorphism from B to B with $\beta|C$ is the transposition. These are prescribed by

$$\delta_i(A) = f_i \text{ for } i = 1, 2, 3.$$

Now $[B, B]_{\mathcal{C}}$ contains four non isomorphisms also. With each of these as the B-component we have exactly one normal cone and thus there are

$$10 \text{ normal cones with vertex } B.$$

Similarly there are 10 normal cones with vertex A and 8 normal cones with vertex C. Thus we have

$$28 \text{ normal cones in this } \mathcal{C}.$$

3 Normal category of semigroups

The category of Principal Left Ideals $\mathcal{L}(S)$ of a regular semigroup S is with right translations as morphisms is usually referred to as the normal category of the semigroup S. Towards the end of this section we show that every normal category arises as the category $\mathcal{L}(S)$ of principal left ideals of a regular semigroup S.

For a regular semigroup S it is known that every principal left ideal is generated by an idempotent. So the set of all principal left ideals can be realised as

$$\{Se : e \text{ is an idempotent}\}.$$

A morphism from Se to Sf in $\mathcal{L}(S)$ is a right translation ρ_u induced by an element $u \in eSf$ and is denoted by

$$\rho(e, u, f) : Se \to Sf \text{ defined by } x \mapsto xu$$

for all $x \in Se$.

The following can be observed.

- For every $x \in Se$ and $u \in eSf$, $xu \in Sf$.
- $\rho(e, u, f) : e \mapsto u$ since $eu = u$.
- $\rho(e, u, f) = \rho(e', u'.f')$ if and only if $e \mathrel{\mathcal{L}} e'$, $f \mathrel{\mathcal{L}} f'$ and $u' = e'u$.
- The identity from Se to Se is $\rho(e, e, e)$

- The compositions are defined by

$$\rho(e,u,f)\rho(f,v,g) = \rho(e,uv,g).$$

The normal category structure on $\mathcal{L}(S)$ is provided by considering the partial order on $v\mathcal{L}(S)$ as usual inclusion and inclusion morphism as usual inclusion mapping.

We observe that if $Se, Sf \in v\mathcal{L}(S)$ and $Se \subseteq Sf$ then $\rho(e,e,f)$ is a morphism in $\mathcal{L}(S)$ and $\rho(e,e,f)$ maps

$$x \mapsto xe = x \text{ for all } x \in Se.$$

Thus the usual inclusion from Se to Sf is a morphism in $\mathcal{L}(S)$.

We proceed to show that $\mathcal{L}(S)$ is a normal category.

Proposition 3. *Every inclusion from Se to Sf in $\mathcal{L}(S)$ has a right inverse in $\mathcal{L}(S)$.*

Proof. Since $Se \subseteq Sf$ we see that $g = fe$ is an idempotent and $Sg = Se$. Also $g = fe \in fSe$ and so $\rho(f,g,e) : Sf \to Se$ in $\mathcal{L}(S)$. Now for every $x \in Se$

$$x(\rho(e,e,f)\rho(f,g,e)) = (xe)g = xefe = xe = x.$$

Thus $\rho(f,g,e)$ is a right inverse of the inclusion $\rho(e,e,f)$. $\square$

It may be observed that the retraction $\rho(f,g,e)$ given above can also be represented as $\rho(f,g,g)$ since $Se = Sg$ and $g \in fSg$.

Now we show that every morphism in $\mathcal{L}(S)$ admits a normal facorization. We use the natural partial order ω on idempotents of a regular semigroup defined by

$$e\; \omega\; f \text{ if } ef = e = fe.$$

Also for any subset A of S we denote by $E(A)$ the set of idempotents in A. Further R_u, L_u etc. denote the green's $\mathcal{R}$, $\mathcal{L}$-classes.

Proposition 4. *Every morphism in $\mathcal{L}(S)$ has a normal factorization and every normal factorization of $\rho(e,u,f)$ is of the form*

$$\rho(e,u,f) = \rho(e,g,g)\rho(g,u,h)\rho(h,h,f) \tag{1}$$

where $h \in E(L_u)$ and $g \in E(R_u) \bigcap \omega(e)$. Here $\rho(e,g,g)$ is a retraction, $\rho(g,u,h)$ is an isomorphism and $\rho(h,h,f)$ is an inclusion.

When S is a regular semigroup $E(R_u) \bigcap \omega(e) \neq \emptyset$ for all $u \in eSf$. Also it can be seen that whenever $u \in R_g \cap L_h,\ \rho(g,u,h)$ is an isomorphism from Sg to Sh. Therefore $\mathcal{L}(S)$ is a category with normal factorization.

3.1 Normal Cones in $\mathcal{L}(S)$

Let S be a regular semigroup and $\mathcal{L}(S)$ be the category of principal left ideals of S. We show that several normal cones exist in $\mathcal{L}(S)$.

For each $a \in S$ we describe a normal cone ρ^a in $\mathcal{L}(S)$ with vertex $Sf = Sa$ as follows. For each $Se \in v\mathcal{L}(S)$

$$\rho^a(Se) = \rho(e, ea, f).$$

Now if $Sg \subseteq Se$ then

$$\rho^a(Sg) = \rho(g, ga, f) = \rho(g, g, e)\rho(e, ea, f)$$

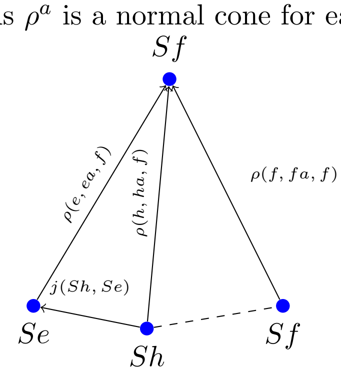
since $gea = ga$ as $ge = g$. That is $\rho^a(Sg) = j(Sg, Se)\rho^a(Se)$. Choosing an idempotent h such that $h\mathcal{R}a$ we see that

$$h \;\mathcal{R}\; a \;\mathcal{L}\; f$$

and so $\rho(h, a, f)$ is an isomorphism. Since $h\mathcal{R}a$ we have $ha = a$ and so

$$\rho^a(Sh) = \rho(h, ha, f) = \rho(h, a, f)$$

is an isomorphism. Thus ρ^a is a normal cone for each $a \in S$.



Choosing $a = f$ we see that the normal cone ρ^f has the propery that

$$\rho^f(Sf) = \rho(f, f, f)$$

is the identity at Sf.

Normal cones of the form ρ^a are usually called **principal cones.** It may be observed that all normal cones in $\mathcal{L}(S)$ may not be principal cones. We give an example of $\mathcal{L}(S)$ for a semigroup S with four elements as given below.

Example 13. *$S = \{a, b, c, d\}$ which is a union of a right zero semigroup $\{a, b\}$ and a left zero semigroup $\{c, d\}$ with $c \leq a$ and $c \leq b$. As a semilattice of rectangular bands this will appear as follows.*

$$\begin{array}{|c|c|}\hline a & b \\ \hline \end{array}$$
$$\begin{array}{|c|}\hline c \\ \hline d \\ \hline \end{array}$$

The semilattice of rectangular bands structure gives that

$$ab = b,\ ba = a,\ ac = ca = c,\ bc = cb = c,\ ad = c = bd,\ da = d = db.$$

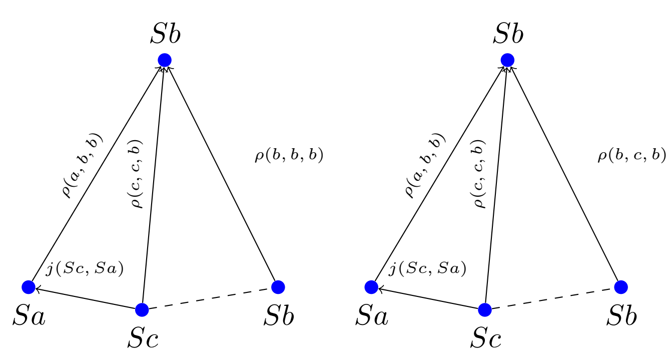
The normal category $\mathcal{L}(S)$ in this case has three objects

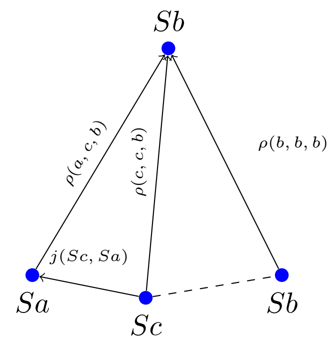
$$Sa = \{a, c, d\},\ Sb = \{b, c, d\},\ Sc = Sd = \{c, d\}.$$

Now we describe the normal cones in this category. First consider a normal cone γ with vertex Sb. The morphisms with codomain Sb are as follows.
Now $aSb = \{b, c\}$ and so
$[Sa, Sb]_{\mathcal{L}(S)} = \{\rho(a, b, b), \rho(a, c, b)\}$.
Similarly $bSb = \{b, c\}$ and so
$[Sb, Sb]_{\mathcal{L}(S)} = \{\rho(b, b, b), \rho(b, c, b)\}$ and $cSb = \{c\}$ and so
$[Sc, Sb]_{\mathcal{L}(S)} = \{\rho(c, c, b)\}$.

Now $\gamma(Sa)$ can be any one of $\{\rho(a, b, b), \rho(a, c, b)\}$. Let $\gamma(Sa) = \rho(a, b, b)$. Then $\gamma(Sb)$ is any one of $\{\rho(b, b, b), \rho(b, c, b)\}$.

Now if $\gamma(Sa) = \rho(a, c, b)\}$ then $\gamma(Sa)$ is not an isomorphism and so $\gamma(Sb)$ needs to be an isomorphism. So there is only one choice $\gamma(Sb) = \rho(b, b, b)$. Thus there are three normal cones with vertex Sb.





Similarly there are three normal cones with vertex Sa and a single one with vertex Sc.

4 Semigroup of normal cones

Now we proceed to show that the set of all normal cones in a normal category forms a semigroup which is a regular semigroup. We usually denote the semigroup of normal cones in $\mathcal{C}$ by $T\mathcal{C}$. With $S = T\mathcal{C}$ we can form the normal category $\mathcal{L}(S)$ and it will turn out that in this case $\mathcal{L}(S)$ is isomorphic to the normal category $\mathcal{C}$. Thus we may say that every normal category is essentially a category of principal left ideals of a regular semigroup.

Towards introducing a multiplication in the set of normal cones we provide two lemmas which are useful in this context. The first lemma shows that the requirement in the definition of normal cones that at least one component is an isomorphism can be replaced by the condition that at least one component is an epimorphism.

Lemma 1. *Let γ be a cone in a normal category $\mathcal{C}$ such that $\gamma(a)$ is an epimorphism for some $a \in v\mathcal{C}$. Then there is $b \in v\mathcal{C}$ such that $\gamma(b)$ is an isomorphism.*

Proof. Let c be the vertex of γ. Since $\gamma(a)$ is an epimorphism we see that the normal factorization of $\gamma(a)$ is

$$\gamma(a) = qu$$

for a retraction $q : a \to b$ and an isomorphism $u : b \to c$ where $b \in v\mathcal{C}$ and $b \leq a$. Now

$$\gamma(b) = j(b,a)\gamma(a) = j(b,a)qu = u.$$

Thus $\gamma(b)$ is an isomorphism. $\square$

The next lemma provides a construction of new normal cones from given ones.

Lemma 2. *Let γ be a normal cone in a normal category $\mathcal{C}$. For any epimorphism $g : c_\gamma \to c$ there is a normal cone $\gamma * g$ in $\mathcal{C}$ whose components are*

$$(\gamma * g)(a) = \gamma(a)g.$$

Proof. First we show that for $b \leq a$ in vC

$$(\gamma * g)(b) = j(b, a)(\gamma * g)(a).$$

Now

$$\begin{aligned} j(b, a)(\gamma * g)(a) &= j(b, a)(\gamma(a)g) \\ &= \gamma(b)g = (\gamma * g)(b). \end{aligned}$$

Since γ is a normal cone there is $b \in v\mathcal{C}$ such that $\gamma(b)$ is an isomorphism. Now

$$(\gamma * g)(b) = \gamma(b)g$$

and is an epimorphism. So by the above lemma $\gamma * g$ is a normal cone. □

Theorem 3. *Let $\mathcal{C}$ be a normal category and $T\mathcal{C}$ be the set of all normal cones in $\mathcal{C}$. For $\gamma, \delta \in T\mathcal{C}$ let $\gamma\delta$ be the normal cone defined by*

$$\gamma\delta = \gamma * (\delta(c_\gamma))^o$$

where c_γ is the vertex of γ and the notation ^{o} denotes the epimorphic component. Then $T\mathcal{C}$ with this product is a regular semigroup.

Proof. By the previous lemma we see that $\gamma\delta$ is a normal cone.
Let $\gamma, \delta, \sigma \in T\mathcal{C}$. Let b be the vertex of $\gamma\delta$. Then $\gamma\delta = \gamma * (\delta(c_\gamma))^o$ so that $b \leq c_\delta$ and $\delta(c_\gamma) = (\delta(c_\gamma))^o j(b, c_\delta)$. Then for any $a \in v\mathcal{C}$

$$\begin{aligned} ((\gamma\delta)\sigma)(a) &= (\gamma\delta)(a)(\sigma(b))^o \\ &= (\gamma(a)(\delta(c_\gamma))^o)(\sigma(b))^o \\ &= (\gamma(a)(\delta(c_\gamma))^o(j(b, c_\delta)\sigma(c_\delta))^o \\ &= (\gamma(a)(\delta(c_\gamma)(\sigma(c_\delta))^o \\ &= \gamma(a)((\delta\sigma)(c_\gamma))^o \\ &= (\gamma(\delta\sigma))(a) \end{aligned}$$

Therefore

$$(\gamma\delta)\sigma = \gamma(\delta\sigma)$$

and so $T\mathcal{C}$ is a semigroup. Now we prove that every normal cone γ has a generalized inverse. Let $\gamma \in T\mathcal{C}$ and let $c_\gamma = c$. Choose a normal cone σ such that $\sigma(c) = 1_c$.
Choose b such that $\gamma(b) = u$ is an isomorphism. Let $\delta = \sigma * u^{-1}$. Then $\delta(c) = u^{-1}$ and vertex of δ is b. Then for any $a \in v\mathcal{C}$

$$\begin{aligned}
(\gamma\delta\gamma)(a) &= \gamma(a)((\delta\gamma)(c))^o \\
&= \gamma(a)(\delta(c)\gamma(b))^o \\
&= \gamma(a)(u^{-1}u)^o \\
&= \gamma(a).
\end{aligned}$$

That is $\gamma\delta\gamma = \gamma$ and so $T\mathcal{C}$ is regular. □

The idempotents in the semigroup $T\mathcal{C}$ have a simple characterization as in the followiing theorem.

Theorem 4. *Let $\gamma \in T\mathcal{C}$ be a normal cone with vertex c. Then γ is an idempotent in the semigroup $T\mathcal{C}$ if and only if*

$$\gamma(c) = 1_c.$$

It may be noted that if $\mathcal{C} = \mathcal{L}(S)$ for a regular semigroup S then $T\mathcal{C}$ may not be isomorphic to S. An easy example is the four element band considered earlier.

Example 14. *$S = \{a, b, c, d\}$ which is a union of a right zero semigroup $\{a, b\}$ and a left zero semigroup $\{c, d\}$ with $c \leq a$ and $c \leq b$ which is a semilattice of rectangular bands.*

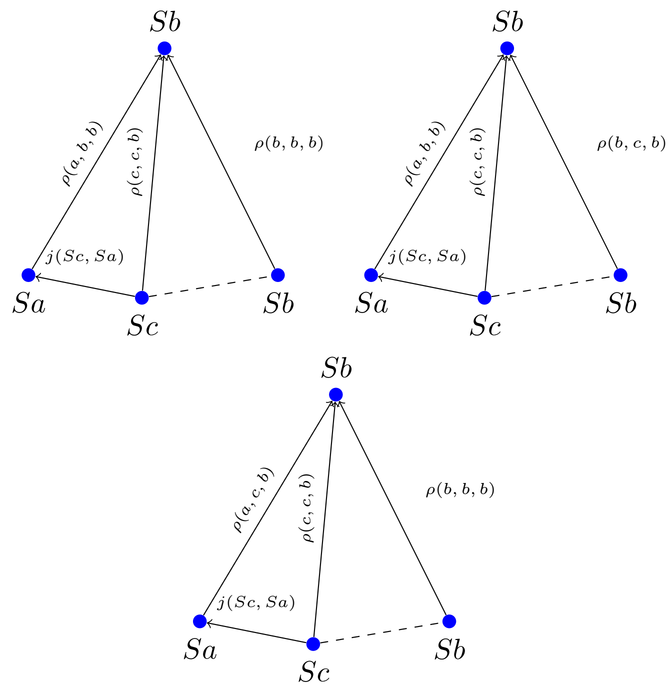
$$\begin{array}{|c|c|}\hline a & b \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline c \\ \hline d \\ \hline \end{array}$$

The normal category $\mathcal{L}(S)$ in this case has three objects

$$Sa = \{a, c, d\},\ Sb = \{b, c, d\},\ Sc = Sd = \{c, d\}.$$

We have seen that there are seven normal cones here. The three normal cones with vertex Sb are the following.



It can be seen that two of the above normal cones are idempotents in $T\mathcal{C}$. Similarly there are two idempotent normal cones with vertex Sa and the single one with vertex Sc is also an idempotent.

Here $T\mathcal{C}$ has seven elements and five of them are idempotents. Clearly $T\mathcal{C}$ is not osomorphic to S.

The following theorem shows that every normal category is essentially a category of principal left ideals of a regular semigroup.

Theorem 5. *Let $\mathcal{C}$ be a normal category and let $S = T\mathcal{C}$. Then $\mathcal{C}$ is isomorphic to $\mathcal{L}(S)$.*

Proof. Describe a functor $\psi : \mathcal{L}(S) \to]categc$ as follows. For $S\epsilon, S\delta \in v\mathcal{L}(S)$ and $\rho = \rho(\epsilon, \sigma, \delta) : S\epsilon \to S\delta$

$$\psi(S\epsilon) = c_\epsilon \text{ and } \psi(\rho) = \sigma(c_\epsilon)j(c_\sigma, c_\delta).$$

Then ψ is an isomorphism of the categories $\mathcal{L}(S)$ and $\mathcal{C}$. □

5 Normal Dual and Cross Connections

A cross connection is a relation connecting two normal categories so that one is isomorphic to the normal category $\mathcal{L}(S)$ and the other is isomorphic to the normal category $\mathcal{R}(S)$ of a regular semigroup S. That is a cross connection determines a regular semigroup S with the above properties. The cross connection relation is provided in terms of two functors each from one category into a special dual called normal dual of the other. The usual dual of a category $\mathcal{C}$ is the category of all homfunctors from $\mathcal{C}$ to the category $\mathcal{S}et$ of sets. For normal dual we consider functors called H-functors which are also set valued functors.

We begin with the definition of H-functors.

Definition 8. *Let $\mathcal{C}$ be a normal category and γ be a normal cone in $\mathcal{C}$. Then the H-functor $H(\gamma, -) : \mathcal{C} \to \mathcal{S}et$ is defined as follows.*

For $a, b \in v\mathcal{C}$ and $g : a \to b$ in $\mathcal{C}$

$$H(\gamma, -)(a) = H(\gamma, a) = \{\gamma * f^o : f : c_\gamma \to a\}$$

and

$$H(\gamma, g) : H(\gamma, a) \to H(\gamma, b) \textit{ is defined by}$$
$$\gamma * f^o \mapsto \gamma * (fg)^o.$$

We can see that each H-functor $H(\gamma, -)$ is naturally isomorphic to the homfunctor $Hom(c, -)$ where $c = c_\gamma$ is the vertex of γ. In fact

$$\eta : H(\gamma, -) \to Hom(c, -)$$

with

$$\eta_a : H(\gamma, a) \to Hom(c, a) \text{ given by } \gamma * f^0 \mapsto f$$

is a natural isomorphism.

Definition 9. *Let $\mathcal{C}$ be a normal category. Then the normal dual of $\mathcal{C}$ is the category $\mathcal{N}^*\mathcal{C}$ of all H-functors with natural transformations as morphisms.*

It can be seen that every natural transformation from $H(\gamma, -)$ to $H(\delta, -)$ is of the form $H(g, -)$ for a morphism $g : c_\delta \to c_\gamma$. That is,

$$H(g, c) : \gamma * f^o \mapsto \delta * (gf)^o$$

for all $f : c_\gamma \to c$. The next theorem gives that $N^*\mathcal{C}$ is a normal category whenever $\mathcal{C}$ is a normal caegory.

Theorem 6. *Let $\mathcal{C}$ be a normal category. Then $\mathcal{N}^*\mathcal{C}$ is also a normal category with partial order given by*

$$H(\gamma, -) \leq H(\delta, -) \textit{ if } H(\gamma, a) \subseteq H(\delta, a) \textit{ for all } a \in v\mathcal{C}.$$

Another concept we use in defining cross connection is that of local isomorphism.

For normal categories $\mathcal{C}$ and $\mathcal{D}$, a functor $\Gamma : \mathcal{C} \to \mathcal{D}$ is said to be a local isomorphism if it is full, faithful, order preserving on objects and is an isomorphism on ideals $\langle c \rangle$ for each $c \in v\mathcal{C}$. Here the ideal $\langle c \rangle$ of $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ with object set $\{a \in v\mathcal{C} : a \leq c\}$.

Now we give the definition of cross connection between normal categories.

Definition 10. *[8] Let $\mathcal{C}$ and $\mathcal{D}$ be normal categories. A cross connection is a pair (Γ, Δ) where $\Gamma : \mathcal{D} \to \mathcal{N}^*\mathcal{C}$ and $\Delta : \mathcal{C} \to \mathcal{N}^*\mathcal{D}$ are local isomorphisms satisfying the condition*

$$c \in M\Gamma(d) \Longleftrightarrow d \in M\Delta(c)$$

for all $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$.

We sometimes denote a cross connection (Γ, Δ) by $(\mathcal{C}, \mathcal{D}, \Gamma, \Delta)$ to specify the categories involved also.

The cross connection semigroup $S(\Gamma, \Delta)$ associated with a cross connection (Γ, Δ) is described as follows. The functors $\Gamma : \mathcal{D} \to \mathcal{N}^*\mathcal{C}$ and $\Delta : \mathcal{C} \to \mathcal{N}^*\mathcal{D}$ are realized as functors from the product category $\mathcal{C} \times \mathcal{D}$ to $\mathcal{S}et$ by defining

$$\Gamma(c, d) = (\Gamma(d))(c) \text{ and } \Delta(c, d) = (\Delta(c))(d).$$

The local isomorphism property of Γ and Δ induces a natural isomorphism $\chi : \Gamma \to \Delta$ where Γ and Δ are functors from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{S}et$. The cross connection semigroup is the following.

$$S(\Gamma, \Delta) = \{(\gamma, \delta) : \gamma \in \Gamma(c, d) \text{ and } \delta = \chi_{(c,d)}(\gamma) \in \Delta(c, d)\}$$

for some $(c, d) \in v\mathcal{C} \times v\mathcal{D}$.

The semigroup structure of $S(\Gamma, \Delta)$ arises as a subsemigroup of $T\mathcal{C} \times T^o\mathcal{D}$ where $T^o\mathcal{D}$ is the semigroup on the set $T\mathcal{D}$ of normal cones in $\mathcal{D}$ with the dual of the usual composition as product. Moreover when $(\gamma, \delta) \in S(\Gamma, \Delta)$ we say that (γ, δ) is a linked pair.

Remark 3. *It is possible that a given γ belongs to $\Gamma(c, d)$ and to $\Gamma(c', d')$ for $c \neq c'$ or $d \neq d'$. Then we may get (γ, δ) and (γ, δ') as linked pairs corresponding to the same γ.*

The next theorem provides a structure theorem for regular semigroups in terms of cross connection.

Theorem 7. *Let S be a regular semigroup. Let $\mathcal{C} = \mathcal{L}(S)$ and $\mathcal{D} = \mathcal{R}(S)$ be the associated normal categories of principal left ideals and principal right ideals respectively. Then there is a cross connection (Γ, Δ) such that the cross connection semigroup $S(\Gamma, \Delta)$ is isomorphic to S.*

Proof. In this case the cross connection (Γ, Δ) is given as follows. For $eS \in v\mathcal{R}(S)$ and $\lambda(e, x, f) : eS \to fS$ in $\mathcal{R}(S)$

$$\Gamma(eS) = H(\rho^e, -) \text{ and } \Gamma(\lambda(e, x, f)) = H(\rho(f, x, e), -)$$

and for $Se \in v\mathcal{L}(S)$ and $\rho(e, y, f) : Se \to Sf$ in $\mathcal{L}(S)$

$$\Delta(Se) = H(\lambda^e, -) \text{ and } \Delta(\rho(e, y, f)) = H(\lambda(f, y, e), -).$$

We end up by getting that in this case

$$S(\Gamma, \Delta) = \{(\rho^a, \lambda^a) : a \in S\}$$

which is isomorphic to S. $\square$

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