IMRT WORKSHOP
FOUNDATIONS OF TOPOLOGY

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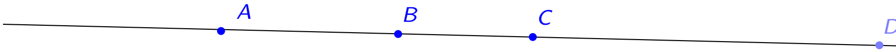
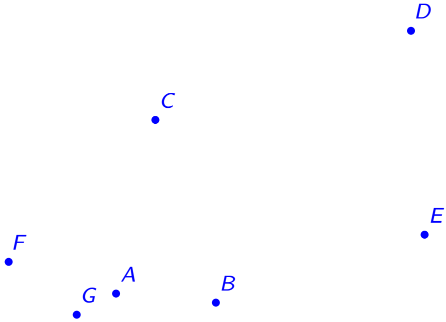
Session 1. Continuity and Neighbourhoods

Consider the points A, B, C, D in the line as given below. Which points are in a neighbourhood of A.


A
B
C
D

Which points in the plane are in a neighbourhood of A.



Consider all rational points on this line.

Which points are in a neighbourhood of 1.

Consider all real points on this line.

Which points are in a neighbourhood of 1.

Which functions are continuous.

For example

$$f(x) = x \sin(1/x) \text{ and } f(0) = 0$$

at $x = 0$.

Check whether $f(x)$ is **close** to 0 whenever x is **close** to 0.
Here **close** means in a neighbourhood.

As another example:

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ otherwise.} \end{cases}$$

Verify continuity at $x = 1$.

Check whether $f(x)$ is **close** to 0 whenever x is **close** to 1.

Which sequences are convergent.

For example

Is

$$(x_n) = (\frac{n+1}{2n+1})$$

convergent.

Does

$$(x_n) = (\frac{n+1}{2n+1})$$

converge to 1, or to 0 or to $1/2$ etc.

Theorem 1 (Bolzano-Weierstrass)

Every bounded infinite set of reals has a limit point.

Cantor [1872] introduced limit points in the framework of classical analysis.

> *By a limit point of a point set P I understand a point situated in a straight line in such a way that every neighbourhood of the point contains infinitely many points of P.*
> *By the "neighbourhood of a point" is meant any interval which contains the point in its interior.*

Closed sets entered into literature at this point and they were sets which contain all its limit points.
Another view of closed sets was that they are sets containing their boundary.

For example

Closed intervals $[a, b]$

$\{1/n : n \in \mathbb{N}\} \cup \{0\}$

Unit circle $\{(x, y) : x^2 + y^2 = 1\}$ in the plane etc.

Open sets entered into literature with Baire's observation.

> *S be an open sphere of radius r. Given any point of S there is a sphere of positive radius with this point as centre, all whose points belong to S. Sets possessing this property I call* **open domains***.*

Caratheodory observed

> *This duality beween closed sets and those which contains purely of interior points, is as we will see, a deep one. If we characterise sets which posses only interior points, we wish to name them as "open".*

Hausdorff introduced axioms for neighbourhood systems.
It is a set E together with a collection S of subsets of E, called neighbourhoods satisfying the following axioms.

A. Every point x belongs to at least one neighbourhood and every neighbourhood of x contains x.
B. If U, V are neighbourhoods of x then there is some neighbourhood W of x such that $W \subseteq U \cap V$.
C. If y belongs to a neighbourhood U of x then there is some neighbourhood V of y such that $V \subseteq U$.
D. If x and y are distinct points then there is a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint.

Clearly $\mathbb{R}$ together with the collection of all open intervals satisfies all the above axioms.

Check the following cases.

1. $\mathbb{R}$ together with the collection of all open intervals each of length 1.
2. $\mathbb{R}$ together with the collection of all closed intervals of positive length.
3. $\mathbb{R}$ together with the collection of all subsets whose complements are finite.
4. $\mathbb{R}$ together with the collection $\{\mathbb{R}, \emptyset\}$.
5. $\mathbb{R}$ together with the collection of all open intervals of the form

$$(a, \infty) : a \in \mathbb{R}.$$

In a metric space (X, ρ), open sets are those which contain an open sphere centered at each of its points.

Theorem 2

The open sets in a metric space X have the following properties

1. *Any union of open sets is open*
2. *Any finite intersection of open sets is open*
3. ϕ *and* X *are open.*

Now let X be a set and S be a neighbourhood system. We may define open sets as follows.
A subset G of X is open if for each $x \in G$ there is a neighbourhood U of x such that $U \subseteq G$.
Verify whether these open sets also satisfy all the properties in the above theorem.

Kuratovski's Closure Spaces

This is considered to be the first version of topological spaces. These are sets X with a mapping $k : P(X) \to P(X)$ stisfying the following axioms called closure axioms.

1. $k(A \cup B) = k(A) \cup k(B)$ for every $A \subset X$.
2. For every subset A of $X, \ A \subseteq k(A)$.
3. $k(\emptyset) = \emptyset$.
4. $k(k(A)) = k(A)$ for all $A \subseteq X$.

Here open sets are defined as complements of closed sets.
Closed sets are A such that $k(A) = A$.
Qn. Veify that these open sets also satisfy the properties in the earlier theorem.

For example

$$A \mapsto A \cup D(A)$$

where $D(A)$ is the set of all limit points of A is a closure operation in the set of reals.
Also

$$A \mapsto A$$

is a closure operation on any set.
So is

$$A \mapsto X$$

for any nonempty subset A of X.

Qn. Verify whether the following is a closure operation on the set of all reals. For any nonempty set A,

$$A \mapsto (-\infty,\ a] \text{ if } A \text{ is bounded above}$$

where a is the supremum of A and

$$A \mapsto \mathbb{R}$$

otherwise.

Qn. Verify whether the following is a closure operation on the set of all reals. For any nonempty set A,

$$A \mapsto A \cup \mathbb{Q}.$$

Here $\mathbb{Q}$ is the set of all rationals.

Qn. Describe all closed sets in the above case.

For example is $[0, 1]$ closed.

Is the set of all irrationals closed.

The modern description of topological space started with Bourbaki, Kelley etc.
A topological space is a pair (X, τ) where X is a nonempy set and τ is a collection of subsets of X called open sets satisfying the following axioms.

1. Any union of open sets is open
2. Any finite intersection of open sets is open
3. ϕ and X are open

Given a topological space (X, τ) we can define neighbourhood system, closue operation, limits etc.
For example neighbourhood of a point x may be defined as any open set containing x.
Given a topological space (X, τ), a neighbourhood system for X can be τ itself.
Also any base is a neighbourhood system which can generate the given toplogy.

Once neighbourhoods are described limits can be taken.
Limit point of a subset A of X is a point $z \in X$ such that A meets each neighbourhood of z.

Closure of a set A in a topological space may be defined as

$$k(A) = \cap\{B : A \subseteq B \text{ and } B \text{ is closed }\}.$$

Or

$$k(A) = A \cup D(A)$$

where $D(A)$ is the set of all limit points of A.

For example let τ be the cofinite topology on the reals.

Qn. Find all limit points of the set $\mathbb{N}$ of naturals.

Qn. Find all closed sets in this space.

Qn. Find the closure of $[0, 1]$ in this space.

Dense and Nowhere dense Sets

(X, τ) be a topological space and $A \subseteq X$.

A is **dense** in X if the closue of A is X.

A is **Nowhere dense** in X if $cl(A)$ has empty interior.

Interior of A is the union of all open sets contained in A.

Qn. Find which of the following are dense and which are nowhere dense in $\mathbb{R}$.

1. $A = \{n, 1/n : n \in \mathbb{N}\}$

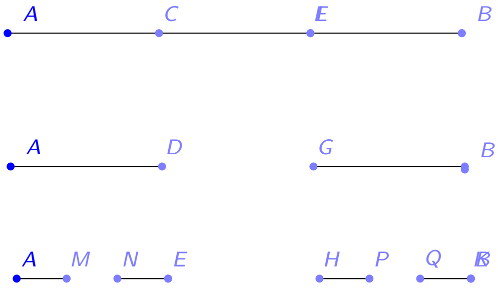
2.$\mathbb{Q}$

3. The set of all irrationals.

4.$\mathbb{Z}$

Qn. Is it true that all nowhere dense sets of reals have empty set of limit points.

Cantor set



Continuity

Continuous functions have been defined for real valued functions of a real variable in classical analysis.
Topological spaces give a general setting for defining coninuity.

Definition 3

Let X, Y be topological spaces. A function $f : X \to Y$ is said to be continuous at $x \in X$ if for every neighbourhood V of $f(x)$ **there is a neighbourhood** U **of** x *such that $f(U) \subseteq V$.*

Equivalenly

$$f^{-1}(V) \text{ contains } U.$$

Or $f^{-1}(V)$ is a neighbourhood of x.

This will lead to the usual definition.

Definition 4

Let X, Y be topological spaces. A function $f : X \to Y$ is said to be continuous for every open set V in Y, $f^{-1}(V)$ is open in X.

Qn. Find all continuous mappings from $\mathbb{R}$ to $\mathbb{Q}$ where $\mathbb{R}$ and $\mathbb{Q}$ have usual topology.

$$f(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ 1 \text{ otherwise.} \end{cases}$$

Is it continuous at $x = 0$.

At $x = 1$

At $x = 2$.

Topological Properties

Homeomorphisms are continuous maps with continuous inverses.
Toplogical spaces X and Y are regarded as identical if they are homeomorphic.
For example $(0,1)$ and $(0,2)$ are homeomorphic with usual topology.
Qn. Are $[0,1]$ and $(0,1)$ homeomorphic.
Qn. Are $\mathbb{R}$ and $\mathbb{R}^2$ homeomorphic.

Properties of topological spaces invariant under homeomorphisms are called topological properties.
Compactness, Connectedness, Normality etc.